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A TURBULENT MHD BOUNDARY LAYER IN A TRANSVERSE MAGNETIC FIELD

by

I. P. Ginzburg and L. I. Skurin

FOREIGN TECHNOLOGY DIVISION

1969

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EDITED TRANSLATION

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English pages: 17

Source: Leningrad. Universitet. Uchenyye Zapiski (Leningrad. University. Studies), No. 338, 1968, pp. 59-74.

Translated by: J. Miller/TDBRO-2

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DATA HANDLING PAGE

Field	Value
01-ACCESSION NO.	TT9000243
98-DOCUMENT LOC	
39-TOPIC TAGS	MHD, turbulent boundary layer, transverse magnetic field, incompressible flow, friction coefficient
09-TITLE	A TURBULENT MHD BOUNDARY LAYER IN A TRANSVERSE MAGNETIC FIELD
47-SUBJECT AREA	20
42-AUTHOR/CO-AUTHORS	GINZBURG, I.P.; 16-SKURIN, L.I.
10-DATE OF INFO	-----68
43-SOURCE	LENINGRAD. UNIVERSITET. UCHENYYE ZAPISKI (RUSSIAN)
68-DOCUMENT NO.	FTD-HT-23-03-69
69-PROJECT NO.	72301-78
63-SECURITY AND DOWNGRADING INFORMATION	UNCL, O
64-CONTROL MARKINGS	NONE
97-HEADER CLASN	UNCL
76-REEL/FRAME NO.	1887 1941
77-SUPERSEDES	
78-CHANGES	
40-GEOGRAPHICAL AREA	UR
NO OF PAGES	17
CONTRACT NO.	
X REF ACC. NO.	65-AT8022260
PUBLISHING DATE	94-00
TYPE PRODUCT	TRANSLATION
REVISION FREQ	NONE
STEP NO.	02-UR/2703/68/000/338/0059/0074
ACCESSION NO.	TT9000243

ABSTRACT: The turbulent boundary layer of an MHD-flow over a flat plate is analyzed. The applied magnetic field is assumed to be transverse to the flow, with all induced effects neglected. The shear stress and the flow enthalpy are expressed as quadratic profiles of the boundary layer coordinate y/δ and velocity v_x, respectively. The electric conductivity is assumed constant and the electric field component E_z in the cross flow direction is neglected. Solutions are obtained for the skin friction coefficient, both with and without the pressure gradients, dp/dx, using the momentum integral method. Although the equations are given for a compressible flow, solutions are obtained only for incompressible flows. It is shown that for $dp/dx \neq 0$, the skin friction τ_w is proportional to

$$\left(\frac{1}{N}\right)^a [1 - \exp(-bN)]^a,$$

where N is the magnetic interaction parameter, $\frac{\sigma B_0^2 \nu}{\rho \upsilon_0^2}$ (B_0 - magnetic field, ρ - density, σ - electric conductivity, υ_0 - external flow velocity, ν - viscosity). Orig. art. has: 49 equations and 6 figures.

A TURBULENT MHD BOUNDARY LAYER IN A TRANSVERSE MAGNETIC FIELD

I. P. Ginzburg and L. I. Skurin

In this work we examine the turbulent boundary layer of a conducting fluid on a dielectric plate in a transverse magnetic field. We investigate both nongradient and gradient flow. The basis of the solutions is the semiempirical turbulence theory, and we use the calculating method shown in [1] for a ordinary boundary layer.

Considering that the magnetic Reynolds numbers are small, we can write the equations of the boundary layer in crossed electrical and magnetic fields associated with the plate in the form

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} = 0, \quad (1)$$

$$\rho v_x \frac{\partial u_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} + \sigma B (E_z - v_x B), \quad (2)$$

$$\frac{\partial p}{\partial y} = 0, \quad (3)$$

$$\rho v_x \frac{\partial H}{\partial x} + \rho v_y \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} (q + \tau v_x) + \sigma E_z (E_z - v_x B), \quad (4)$$

$$p = \rho R T, \quad (5)$$

where E_z is the external electrical field strength; B is magnetic induction; σ is conductivity of the fluid; q is the transverse component of the heat flow vector; τ is the xy component of the stress tensor; H is the total enthalpy per unit of mass; the remaining designations are those commonly used.

Considering (3) and bearing in mind that outside the boundary layer $\frac{\partial v_x}{\partial y} = 0$ and $\frac{\partial \tau}{\partial y} = 0$, from (2) we get

$$-\frac{dp}{dx} = \rho_0 u \frac{du}{dx} + \sigma_0 B^2 u - \sigma_0 B E_z, \tag{6}$$

where u is the velocity on the outer boundary of the boundary layer, and the subscript "0" pertains to the outer boundary. From (4), which pertains to external flow, considering that outside the boundary layer $q = \tau = \frac{\partial H}{\partial y} = 0$, we get

$$\rho_0 u \frac{dH_0}{dx} = \sigma_0 E_z (E_z - uB),$$

whence it follows that when $E_z = 0$ the total heat content outside the boundary layer

$$H_0 = c_p T_0 + \frac{u^2}{2} = \text{const.} \tag{7}$$

Examination of (6) leads to various statements of the problem.

1. Nongradient flow $-\frac{dp}{dx} = 0$. In this case the velocity on the outside boundary is determined from the relationship

$$\rho_0 u \frac{du}{dx} + \sigma_0 B^2 u - \sigma_0 B E_z = 0, \tag{8}$$

where the external magnetic and electrical fields should be given. In the particular case $\sigma_0 = 0$, from (8) it follows that u = const.

2. Gradient flow $-\frac{dp}{dx} \neq 0$. In this case the pressure gradient is determined from (6) with given σ_0, u, B, and E_z. Let us examine certain of the simplest cases of the indicated statements of the problem. When solving, the values of τ and q which enter into (2) and (4) will be determined, starting from the semiempirical turbulence theory, using the two-layer scheme of the boundary layer. According to this theory,

$$\text{when } y < \delta_1 \quad \tau = \mu \frac{\partial v_x}{\partial y};\ q = \lambda \frac{\partial T}{\partial y},$$

$$q + \tau v_x = \mu \left[\frac{1}{\Pr} \cdot \frac{\partial H}{\partial y} + \left(1 - \frac{1}{\Pr}\right) v_x \frac{\partial v_x}{\partial y}\right];$$

$$\text{when } y \geqslant \delta_1 \quad \tau = \rho k^2 y^2 \left(\frac{\partial v_x}{\partial y}\right)^2,\ q = \rho k k_T y^2 \frac{\partial v_x}{\partial y} \frac{\partial c_p T}{\partial y},$$

$$q + \tau v_x = \rho k^2 y^2 \frac{\partial v_x}{\partial y} \left[\frac{1}{\Pr_t} \frac{\partial H}{\partial y} + \left(1 - \frac{1}{\Pr_t}\right) v_x \frac{\partial v_x}{\partial y}\right],$$

where δ_1 is the thickness of the laminar sublayer; μ, λ, c_p are,

respectively, the coefficients of dynamic viscosity, thermal conductivity, and heat capacity at constant pressure; T is temperature; $Pr = \frac{c_p \mu}{\lambda}$, $Pr_t = \frac{k}{k_t}$, k, k_t are universal constants of the semi-empirical turbulence theory. Henceforth we will consider Pr and c_p constant and $k = k_t$.

It is assumed that when crossing the boundary of the laminar sublayer the values τ, q, $h = c_p T$, and v_x do not undergo a discontinuity, while

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_1-0} = k_1\left(\frac{\partial v_x}{\partial y}\right)_{y=\delta_1+0}, \quad \left(\frac{\partial h}{\partial y}\right)_{y=\delta_1-0} = k_1 \Pr\left(\frac{\partial h}{\partial y}\right)_{y=\delta_1+0}, \tag{9}$$

where k_1 is a universal constant.

For an approximate solution to system (1)-(5) we set

$$\begin{gathered}\tau = \tau_w + A_1\frac{y}{\delta} + A_2\left(\frac{y}{\delta}\right)^2,\\ H = B_1 + B_2 v_x + B_3 v_x^2 \text{ when } y \leqslant \delta_1;\\ H = C_1 + C_2 v_x \text{ when } y \geqslant \delta_1.\end{gathered} \tag{10}$$

Coefficients A_1, A_2, B_1, B_2, B_3, C_1, and C_2 are determined using the boundary conditions, the motion and energy equations, relationship (9), and the condition τ = 0 when y = δ (δ is the boundary layer thickness).

The solution for such a scheme for a nonconducting fluid is given in [1]. Henceforth we will use the results of this work.

Nongradient flow, $E_z = 0$, σ = const. We have the same boundary conditions as for the boundary layer of an nonconducting fluid:

$$\begin{gathered}y = 0,\ v_x = v_y = 0,\ H = H_w = c_p T_w;\\ y = \delta,\ v_x = u,\ H = H_0,\end{gathered} \tag{11}$$

where, according to (7), H_0 = const. Let us define u. The density on the outer boundary, according to (3), (5), and (7), can be represented in the form

$$\rho_0 = \frac{\rho_w T_w}{T_0} = \frac{c_p T_w}{H_0} \cdot \frac{\rho_w}{1 - \frac{u^2}{2H_0}}. \tag{12}$$

Substituting this expression into (8), where $E_z = 0$, we get an equation for determining the velocity at the outer boundary

$$\frac{c_p T_w}{H_0}\cdot\frac{\rho_w}{1-\frac{u^2}{2H_0}}\cdot\frac{du}{dx}=\sigma B^2.$$

Substituting this expression into (u), where $T_w \rho_w = \text{const}$ (since $p = \text{const}$), from the last equation we get

$$\operatorname{Arth}\bar{u}=\operatorname{Arth}\bar{u}_0-\frac{\sigma B^2}{\rho_w\sqrt{2H_0}}\frac{x}{\bar{H}_w}, \tag{13}$$

where $\bar{u}\equiv\frac{u}{\sqrt{2H_0}}$; $\bar{u}_0\equiv\frac{u_0}{\sqrt{2H_0}}$; $\bar{H}_w\equiv\frac{H_w}{H_0}$.

The integral pulse relationship, corresponding to (2), where $\frac{\partial p}{\partial x} = 0$, $E_z = 0$, with consideration of (8), can be written in the form

$$\frac{d}{dx}(\rho_0 u^2\delta^{**})=\tau_w+\sigma B^2 u\left[\delta^*-\int_0^\delta\left(1-\frac{v_x}{u}\right)dy\right], \tag{14}$$

where

$$\delta^{**}\equiv\int_0^\delta\frac{\rho}{\rho_0}\frac{v_x}{u}\left(1-\frac{v_x}{u}\right)dy,\quad \delta^*\equiv\int_0^\delta\left(1-\frac{\rho}{\rho_0}\frac{v_x}{u}\right)dy. \tag{15}$$

Since in this case $\frac{\partial\tau}{\partial y} = 0$ when $y = 0$, the dependence of τ on y/δ can, approximately, be represented in the form $\tau = \tau_w\left(1 - \frac{y^2}{\delta^2}\right)$, and then for the velocity profile we get the same expression as in [1]; δ^{**} on $\zeta \equiv \frac{u}{\sqrt{\tau_w/\rho_w}}$ will be the same as in the absence of a magnetic field.

Calculation of the integral on the right side of (14), using the velocity profile obtained in [1], gives

$$\int_0^\delta\left(1-\frac{v_x}{u}\right)dy=\frac{\nu_w}{u}\frac{2D}{e}k\zeta_a e^{ak\zeta_a}\left[\frac{d_1\cos d_1+\sin d_1}{d_1^2+1}+\right.$$
$$\left.+\frac{e^{-\frac{k\zeta}{\bar{u}_w}(d_1+d_2)}}{d_2^2+1}(d_2\cos d_2+\sin d_2)\right],$$

where $d_1 = \arcsin\frac{1}{a}\left(1-\frac{\bar{a}}{2\bar{u}^2}\right)$, $d_2 = \arcsin\frac{\bar{a}}{2a\bar{u}^2}$, and, according to [1],

$$\bar{u}_w\equiv\frac{u}{\sqrt{2H_w}},\ D\equiv D(\bar{u},\bar{H}_w,\Pr),\ a\equiv a(\bar{u},\bar{H}_w,\bar{u}_w);$$
$$\alpha\equiv\alpha(\zeta,\bar{u},\bar{H}_w,\Pr),\ \bar{a}\equiv\bar{a}(\zeta,\bar{u},\bar{H}_w,\Pr).$$

The values of ρ_0 and ρ_w as functions of x are determined by relationships (12), (13), (5), and (11). Thus, the problem reduces to numerical integration of (14).

Let us examine an incompressible fluid. Pulse relationship (14) is written in the form

$$\frac{d}{dx}(u^2\delta^{**}) = \frac{\tau_w}{\rho}; \tag{16}$$

the velocity on the outer boundary, according to (13), will be defined by the expression

$$u = u_0(1 - S_x),\ S_x \equiv \frac{\sigma B^2 x}{\rho u_0};$$

the width of the lost pulse, according to [1],

$$\delta^{**} = \frac{k_1}{k^4 e^{k_1}} \frac{\nu}{u} e^{k\zeta}\left(1 - \frac{2}{k\zeta}\right). \tag{17}$$

Setting in this last relationship, approximately, $\left(1 - \frac{2}{k\zeta}\right) = \left(1 - \frac{2}{k\zeta}\right)_{cp} = 0.8$ and substituting (17) into (16), we get the following equation for determining ζ as a function of x:

$$\frac{d}{dx}(ue^{k\zeta}) = \frac{k^4 e^{k_1}}{0{,}8k_1\nu} \frac{u^2}{(k\zeta)^2}. \tag{18}$$

To obtain an approximate analytical solution to (18) let us replace, considering S_x small, u by u_0 on the left side of (18) (the error in determining ζ when $S_x \leq 0.4$ does not exceed 6%). We then get the solution in the form

$$\int_0^{k\zeta} \eta^2 e^{\eta} d\eta = \frac{k^4 e^{k_1}}{0{,}8k_1} \mathrm{Re}_x\left(1 - S_x + \frac{1}{3}S_x^2\right), \tag{19}$$

where

$$\int_0^{k\zeta} \eta^2 e^{\eta} d\eta = e^{k\zeta}(k\zeta - 1)^2 + e^{k\zeta} + 2. \tag{20}$$

Considering $k\zeta$ to be a sufficiently large value, let us disregard, in this last equality, the second and third terms on the right; then, using the approximation

$$\eta^2 e^{\eta} \approx e^{n_1}(e^{n_2\eta} - 1),\ n_1 = \text{const},\ n_2 = \text{const}, \tag{21}$$

we get

$$e^{n_2 k\zeta} = \frac{k^4 \exp(k_1 + n_2 - n_1 - 1)}{0{,}8k_1} \mathrm{Re}_x\left(1 - S_x + \frac{1}{3}S_x^2\right), \tag{22}$$

whence

$$\zeta \approx \frac{1}{n_2 k} \ln\left[\frac{k^4 \exp(k_1 + n_2 - n_1 - 1)}{0{,}8k_1} \mathrm{Re}_x\right] - $$
$$- \frac{1}{n_2 k} S_x = \zeta_{B=0} - \frac{1}{n_2 k} S_x. \tag{23}$$

For values of $k\zeta$ = 6-12, n_1 = 2.15 and n_2 = 1.25. Setting k_1 = 4.29 and k = 0.39, just as for the usual turbulent boundary layer, from the last relationship we get

$$\zeta \approx \zeta_{B=0} - 2{,}0 S_x.$$

In [2], where there is obtained a solution for the examined problem from concepts of the theory of dimensionality, there is given an analogous relationship with the coefficient 1.87.

Having set $v^* = \sqrt{\frac{\tau_w}{\rho}}$, we have

$$\frac{v^*}{v^*_{B=0}} = \frac{v^*}{u} \cdot \frac{u_0}{v^*_{B=0}} \cdot \frac{u}{u_0} = \frac{\zeta_{B=0}}{\zeta}(1 - S_x),$$

whence, using (23), we find

$$\left(1 - \frac{v^*}{v^*_{B=0}}\right)\frac{1}{S_x} = 1 - \frac{0{,}434\,(1 - S_x)}{\lg\left[\frac{k^4 \exp(k_1 + n_2 - n_1 - 1)}{0{,}8k_1} \mathrm{Re}_x\right]}. \tag{24}$$

Figure 1 gives the limiting curve constructed for the case S_x = 0, on the right side of (24). The same figure shows, for comparison, the corresponding curve obtained in [2].

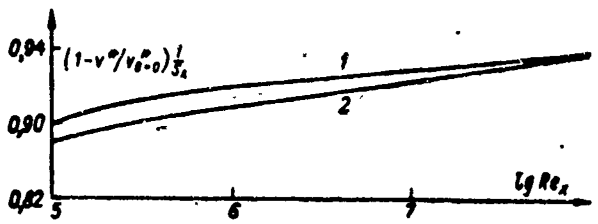

Fig. 1. Rate of decrease of dynamic velocity per unit of parameter $S_x\left(\frac{dp}{dx} = 0, \sigma = \text{const}\right)$. 1 – from data in [2]; 2 – present work.

Let us introduce a formula for the local friction coefficient:

$$c_f = \frac{2\tau_w}{\rho u_0^2} = \frac{2}{\zeta^2}(1 - S_x)^2. \tag{25}$$

From (21), with sufficiently large n, we have

$$\frac{1}{\eta^{3}} = e^{(1-n_2)\eta - n_1}, \tag{26}$$

which, after substitution into (25) and use of (22), gives

$$c_f = \frac{2k_1}{k^2 e^{k_1 n_2}} \left[\frac{k^4 \exp(k_1 + n_2 - n_1 - 1)}{0{,}8k_1} \right]^{\frac{1}{n_2}} \times$$

whence

$$\times \left(1 - S_x + \frac{1}{3} S_x^2\right)^{\frac{1}{n_2} - 1} (1 - S_x)^2 \, \mathrm{Re}_x^{\frac{1}{n_2} - 1},$$

$$\frac{c_f}{c_{f0}} = \left(1 - S_x + \frac{1}{3} S_x^2\right)^{\frac{1}{n_2} - 1} (1 - S_x)^2.$$

In [3], the value $\frac{c_f}{c_{fB=0}} = 1 - 2.69x + \ldots$ was found for the laminar boundary layer. Comparison of the ratios of local friction coefficients for laminar and turbulent boundary layers, given in Fig. 2, shows that the magnetic field has a stronger influence on the laminar layer.

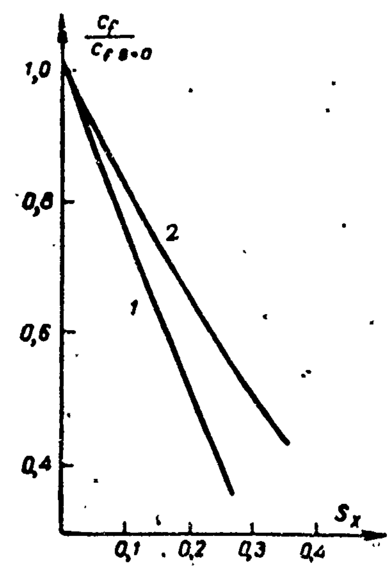

Fig. 2. Comparison of the ratios of local friction coefficients for laminar (1) and turbulent (2) boundary layers $\left(\frac{dp}{dx} = 0,\ \sigma = \text{const}\right)$.

<u>Nongradient flow</u> $E_z = 0$, $\sigma = 0$. As in [4], we use for the conductivity, the following integral relationship:

$$\sigma = \frac{\rho}{\rho_0} \cdot \frac{T_w}{T_0} \left(1 - \frac{v_x}{u}\right) \sigma_w, \tag{27}$$

whence $\sigma_0 = 0$, and then, as a result of (8), the velocity on the outer boundary is constant (u = const). The boundary conditions are given by relationships (11).

The integral pulse relationship corresponding to equation of motion (2), where $\frac{\partial p}{\partial x} = 0$, $E_z = 0$, and σ is determined by (27), will have the form

$$\frac{d\delta^{**}}{dx} - \frac{\sigma_w B^2}{\rho_w u}\delta^{**} = \frac{\tau_w}{\rho_0 u^2}. \qquad (28)$$

The velocity profile in the boundary layer in this case as well can be taken approximately as in [1].

The width of the lost pulse, according to x[1], can be represented approximately by the expression

$$\delta^{**} = \frac{2D}{\varepsilon}\cdot\frac{\rho_w}{\rho_0}\cdot\frac{\nu_w}{u_0}\sqrt{\frac{\overline{H}_w}{1-\overline{u}^2}}\,e^{ak\zeta}.$$

Since $u = u_0$ = const, then T_0 is also constant by virtue of (7). Henceforth we will consider that H_w = const (or T_w = const); then ρ_w = const, $\overline{H}_w$ = const, and $\overline{u}_w$ = const. Having substituted this last formula into (28), and making simple transformations, we get

$$\frac{d}{d\,\mathrm{Re}_x}e^{ak\zeta} - \frac{\sigma_w B^2 \nu}{\rho_w u_0^2}e^{ak\zeta} = \frac{1}{D_1 (ak\zeta)^2}, \qquad (29)$$

where $D_1 \equiv \frac{2D}{a^2k^2\varepsilon}\sqrt{\frac{\overline{H}_w}{1-\overline{u}^2}}$.

In the case of a uniform magnetic field (B = const), equation (29) reduces to the quadrature

$$\int_0^{ak\zeta}\frac{\eta^2 e^{\eta}\,d\eta}{ND_1\eta^2 e^{\eta}+1} = \frac{\mathrm{Re}_x}{D_1},$$

where $S = \frac{\sigma_w B^2 \nu}{\rho_w u_0^2}$.

To obtain a solution in elementary functions, we can use approximation (21).

Let us examine in more detail for the possibility of comparison with the results obtained for the laminar layer, the case when $B = B_0/\sqrt{x}$, B_0 = const. Let us set, in (29),

$$B^2 = \frac{B_0^2 e^{n_2-1}\frac{u_0}{\nu}}{D_1\left(1 - Se^{n_2-1}\right)} \cdot \frac{1}{(ak\zeta)^2 e^{ak\zeta}},$$

where $S \equiv \frac{\sigma_w B_0^2}{\rho_w u_0}$. We get

$$\frac{d}{d\,\mathrm{Re}_x} e^{ak\zeta} = \frac{1}{D_1 (ak\zeta)^2 \left(1 - Se^{n_2-1}\right)}$$

or

$$\int_0^{ak\zeta} \eta^2 e^{\eta} d\eta = \frac{\mathrm{Re}_x}{D_1\left(1 - Se^{n_2-1}\right)},$$

whence, by virtue of (20) and (21), we get

$$(ak\zeta)^2 e^{ak\zeta} = \frac{e^{n_2-1}\mathrm{Re}_x}{D_1\left(1 - Se^{n_2-1}\right)}.$$

This last relationship makes it possible to transform the expression for B^2 to the form $B = \frac{B_0}{\sqrt{x}}$, and is the solution of (29) for the given distribution of the magnetic field.

Using approximation (21) we get

$$e^{n_2 ak\zeta} = \frac{e^{n_2-1-n_2}\mathrm{Re}_x}{D_1\left(1 - Se^{n_2-1}\right)}. \tag{30}$$

From this it follows that when $S \geq e^{1-n_2}$, there is separation of the boundary layer.

Relationship (26), using (30), gives a formula for the local friction coefficient:

$$c_f \equiv \frac{2\tau_w}{\rho u_0^2} = \frac{2}{\zeta^2} = \frac{A}{n_2}\left(\frac{\mathrm{Re}_x}{1 - Se^{n_2-1}}\right)^{\frac{1}{n_2}-1}, \tag{31}$$

where $A = 2k^2 a^2 n_2 \exp\left(-\frac{n_2^2 + n_1 + 1}{n_2}\right) D_1^{1-\frac{1}{n_2}}$. From this the coefficient of friction of a plate of length L

$$c_f \equiv \frac{1}{L}\int_0^L c_f dx = A' \mathrm{Re}_L^{\frac{1}{n_2}-1}\left(1 - Se^{n_2-1}\right)^{1-\frac{1}{n_2}}. \tag{32}$$

Using (27) and (29) it is also easy to calculate the coefficient of the stresses caused by three-dimensional magnetic forces in the boundary layer:

$$C_M \equiv \frac{2}{\rho_0 u_0^2 L}\int_0^L\int_0^\delta \sigma B^2 v_x dy dx = A\,\mathrm{Re}_L^{\frac{1}{n_1}-1}\frac{Se^{n_1-1}}{(1-Se^{n_1-1})^{\frac{1}{n_1}}}.$$

The coefficient of resistance of a plate of length L

$$C_c \equiv C_f + C_M = A\,\mathrm{Re}_L^{\frac{1}{n_1}-1}(1-Se^{n_1-1})^{-\frac{1}{n_1}}.$$

The heat flow on the plate is given, according to [1], by the expression

$$q_w = \frac{H_0 - H_w - \frac{1-\mathrm{Pr}}{2}u_1^2}{2u_0\left[1-(1-\mathrm{Pr})\frac{u_1}{u_0}\right]}c_f\rho_0 u_0^2, \tag{33}$$

where $u_1 = \frac{k_1}{k}u_0\sqrt{\frac{\overline{H}_w}{1-\overline{u}^2}}\sqrt{\frac{c_f}{2}}$. Substituting (31) into (33), we get the distribution of the heat flow along the plate.

In the case Pr = 1, from (33) it is easy to determine the heat flow of a plate of length L

$$Q_w = \frac{1}{L}\int_0^L q_w dx = \frac{H_0 - H_w}{2}\rho_0 u_0 C_f,$$

whence, considering (32),

$$\frac{Q_w}{Q_{wB=0}} = \frac{C_f}{C_{fB=0}} = (1-Se^{n_1-1})^{1-\frac{1}{n_1}}.$$

Figure 3 shows the dependence of the coefficients C_f, C_M, and C_c on parameter S. Figure 4 compares the heat flow ratios with and without a magnetic field for laminar [4] and turbulent boundary layers. As follows from Fig. 4, the magnetic field exerts less influence on the turbulent layer than on the laminar layer.

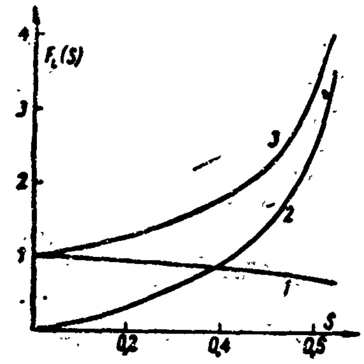

Fig. 3. Coefficient of resistance of the plate $\left(\frac{dp}{dx} = 0,\ \sigma_0 = 0,\ B = \frac{B_0}{\sqrt{x}}\right)$.
$F_1(S) = C_1 \frac{Re_L^{0.2}}{A}$. 1 — $i = f$; 2 — $i = M$; 3 — $i = c$.

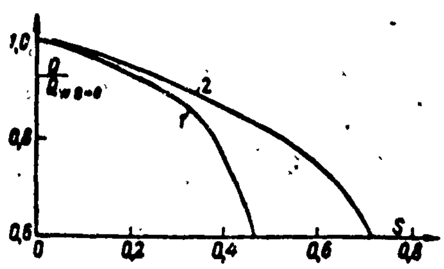

Fig. 4. Comparison of heat flow for laminar (1) and turbulent (2) boundary layers $\left(\frac{dp}{dx} = 0,\ \sigma_0 = 0,\ B = \frac{B_0}{\sqrt{x}}\right)$.

Gradient flow, σ = const. In this case the pressure gradient is determined from (6) by giving the values u, B, and E_z. By virtue of the constancy of the pressure across the boundary layer, we can substitute (6) into (2); then we get

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = \rho_0 \frac{du}{dx} u + \frac{\partial \tau}{\partial y} + \sigma B^2 (u - v_x). \qquad (34)$$

When there is an external electrical field, it is more difficult to determine the parameters of the compressible fluid on the outer boundary since the condition of constant total heat content is not satisfied. If we consider that there is no external electrical field, the temperature and density on the outer boundary are determined from

the condition of constant heat content (7) and the Clapeyron equation (5). From (7) we have

$$T_0 = \frac{H_0}{c_p} - \frac{u^2}{2c_p}, \quad \frac{dT_0}{dx} = -\frac{u}{c_p}\frac{du}{dx}; \tag{35}$$

differentiating (5) with respect to x and using the last relationships, we get

$$\frac{dp}{dx} = \frac{R}{c_p}\frac{d\rho_0}{dx}\left(H_0 - \frac{u^2}{2}\right) - \frac{R}{c_p}\rho_0 u \frac{du}{dx}$$

or, replacing $\frac{dp}{dx}$ from (6) (where $E_z = 0$),

$$\frac{R}{c_p}\left(\frac{u^2}{2} - H_0\right)\frac{d\rho_0}{dx} + \left(\frac{R}{c_p} - 1\right)u\frac{du}{dx}\rho_0 = \sigma_0 B^2 u. \tag{36}$$

Thus, with given u (and also σ_0 and B), the temperature and density on the outer boundary are determined, respectively, by the first of relationships (35) and by relationship (36).

In the case of an incompressible fluid, if the conductivity does not depend on the temperature, the dynamic and thermal problems are separate. In particular, when σ = const, the electrical field does not, according to (34), influence flow in the boundary layer and enters only into (6) which connects the pressure gradient and the velocity on the outer boundary. Let us examine this case in detail.

Coefficients A_1 and A_2 of approximation polynomial (10) for friction are determined from the conditions

$$y = \delta, \; \tau = 0, \; \frac{\partial \tau}{\partial y} = 0$$

(the last equality follows from (34)); then

$$\tau = \tau_w\left(1 - \frac{y}{\delta}\right)^2. \tag{37}$$

Starting with this last relationship and using the propositions of the semiempirical turbulence theory stated previously, we find

$$\text{when } y \leqslant \delta_1, \; \frac{v_x}{u} = \frac{1}{\zeta^2}\frac{uy}{\nu};$$

$$\text{when } y \geqslant \delta_1, \; \frac{v_x}{u} = 1 + \frac{1}{k\zeta}\left(\ln e\frac{y}{\delta} - \frac{y}{\delta}\right);$$

$$\frac{u\delta_1}{\nu} = \frac{k_1}{k}\zeta.$$

Joining the velocity profiles on the boundary of the laminar sublayer we get

$$\frac{u\delta}{\nu} = \frac{k_1}{k^2} e^{1-k_1} k\zeta e^{k\zeta}. \tag{38}$$

The integral pulse relationship, corresponding to (34), is written in the form

$$\frac{u_0}{u}\frac{d\frac{u\delta^{**}}{\nu}}{d\,\mathrm{Re}_x} = \frac{\tau_w}{\rho u^2} - \frac{u_0}{u^2}\frac{du}{d\,\mathrm{Re}_x}\frac{u(\delta^{**}+\delta^*)}{\nu} - \frac{\sigma B^2\nu}{\rho u^2}\frac{u\delta^*}{\nu}, \tag{39}$$

where $\mathrm{Re}_x \equiv \frac{u_0 x}{\nu}$, δ^{**} and δ^* are determined from (15) ($\rho_0 = \rho$).

In view of the small thickness of the laminar sublayer, let us calculate approximately the integrals in (15), using only the velocity profile corresponding to the turbulent part of the boundary layer. Having integrated, and considering (38), we get

$$\frac{u\delta^*}{\nu} = \frac{k_1}{2k^2} e^{1-k_1} e^{k\zeta}, \quad \frac{u\delta^{**}}{\nu} = \frac{k_1}{2k^2} e^{1-k_1} e^{k\zeta}\left(1 - \frac{\delta}{3k\zeta}\right). \tag{40}$$

Let us calculate the constant coefficients in (38) and (39), and the factor in parentheses in (40), which is the ratio δ^{**}/δ^*; let us average, in the proposed range of change, ζ and designate it by $\varkappa$ (in the first approximation we set $\varkappa = 1$). We will have

$$\frac{u\delta}{\nu} = 1{,}05k\zeta e^{k\zeta}, \quad \frac{u\delta^*}{\nu} = \frac{1}{\varkappa}\cdot\frac{u\delta^{**}}{\nu} = 0{,}525e^{k\zeta}. \tag{41}$$

Substituting this into (39), we get, finally, the equation

$$\varkappa\frac{u_0}{u}\cdot\frac{dk\zeta}{d\,\mathrm{Re}_x} = \frac{0{,}29}{(k\zeta)^2 e^{k\zeta}} - \frac{u_0}{u^2}\cdot\frac{du}{d\,\mathrm{Re}_x}(1+\varkappa) - \frac{\sigma B^2\nu}{\rho u^2}, \tag{42}$$

integration of which gives the solution of the problem for given $u(x)$, $B(x)$.

Let us examine certain analytical solutions of (42). When $u = u_0/1 + b\,\mathrm{Re}_x$ and $B = B_0/1 + b\,\mathrm{Re}_x$ ($b \geq 0$, $u_0 = \mathrm{const}$, $B_0 = \mathrm{const}$), equation (42) reduces to the quadrature

$$\int_0^{k\zeta} \frac{\eta^2 e^{\eta} d\eta}{1 + 3{,}45[b(\varkappa+1) - N]\eta^2 e^{\eta}} = \frac{0{,}29}{\varkappa b}\ln(1 + b\,\mathrm{Re}_x), \tag{43}$$

where $N \equiv \frac{\sigma B_0^2 \nu}{\rho u_0^2}$.

When $N > b(\varkappa + 1)$, integral (43) diverges at the point defined by the relationship

$$(k\zeta)^2 e^{k\zeta} = \frac{0{,}29}{N - b(\varkappa+1)}. \tag{44}$$

From this it follows that with sufficient large N the parameters of the boundary layer will have limiting values that are defined by (44). When $N \leq b(\kappa + 1)$, the nature of the change in parameters will be the same as in an ordinary boundary layer.

Let us examine in more detail the case $b = 0$ ($u = u_0 = \text{const}$, $B = B_0 = \text{const}$). Let us set $\kappa = 0.8$ (the proposed range of change of $k\zeta$ is from 6 to 12). Relationship (43) is rewritten in the form

$$\int_0^{k\zeta} \frac{\eta^{\kappa} e^{\eta} d\eta}{1 - 3{,}46 N \eta^{\kappa} e^{\eta}} = 0{,}361\, \mathrm{Re}_x.$$

Integrating using approximation (21) and making the obvious transformations, we get

$$e^{a_1 k\zeta} = \frac{0{,}29}{e^{a_1} N}\left(1 - e^{-1{,}25 a_1 N \mathrm{Re}_x}\right), \qquad (45)$$

whence, as a result of (26),

$$\frac{\tau_w}{\rho u_0^2} = \frac{k^2}{(k\zeta)^2} = \frac{0{,}152}{e^{a_1}}\left(\frac{0{,}29}{e^{a_1} N}\right)^{\frac{1}{a_1}-1}\left(1 - e^{-1{,}25 a_1 N \mathrm{Re}_x}\right)^{\frac{1}{a_1}-1}. \qquad (46)$$

The limiting values of $\frac{\tau_w}{\rho u_0^2}$ as functions of N, calculated from (44) (when $b = 0$), are given in Fig. 5. (Naturally, these same values can be obtained approximately from (46) when $\mathrm{Re}_x \to \infty$.) This same figure gives the limiting values of the dimensionless thickness of the boundary layer, calculated from the formula $\frac{u_0 \delta_{max}}{\nu} = \frac{0.78}{N \zeta_{max}}$, which follows from (41) and (44). For comparison, Fig. 5 gives the corresponding curves obtained in [5] using the velocity profile that was determined semiempirically by Harris [6].

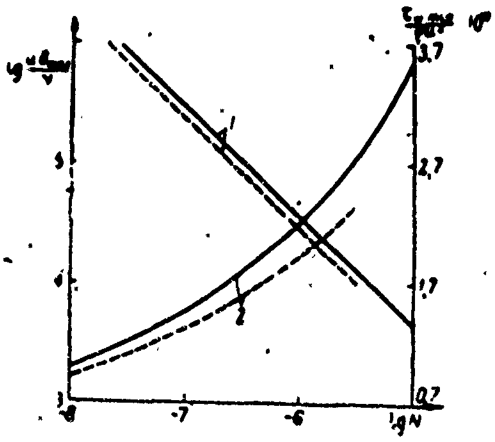

Fig. 5. Limiting values of surface friction and boundary layer thickness $\left(\frac{dp}{dx} \neq 0,\ u = \text{const},\ B = \text{const}\right)$. 1 – $\lg\frac{u\delta_{max}}{\nu}$; 2 – $\frac{\tau_{w\,min}}{\rho u^2}$. Solid line: data from this article; dashed line: data from [5].

The presently examined case corresponds, approximately, to flow on the dielectric wall of the inlet section of an MHD channel with parallel walls. With a sufficiently large value of parameter N, the boundary layers on the opposite walls of the channel do not join. In this case the friction across the channel in the section of developed flow will be determined, according to (37), by the relationship

$$\frac{\tau}{\tau_w}=\left(1-\frac{y}{h}\cdot\frac{\mathrm{Re}_h}{\frac{u_0\delta_{max}}{\nu}}\right)^2,$$

where h is half the height of the channel; $\mathrm{Re}_h \equiv \frac{u_0 h}{\nu}$. Substituting here $\frac{u_0\delta_{max}}{\nu}$ and introducing the designations $M^2 \equiv \frac{\sigma B^2 h^2}{\rho\nu}$, $R^* \equiv \frac{v^* h}{\nu}$, we get

$$\frac{\tau}{\tau_w}=\left(1-1{,}28\frac{y}{h}\frac{M^2}{R^*}\right)^2.$$

From this formula it follows that there is no friction in the channel when $\frac{M^2}{R^*}\frac{y}{h} > 0.78$ (Harris [6] obtained $\frac{M^2}{R^*}\cdot\frac{y}{h} > 0.6$).

For a more precise solution of the problem of the boundary layer in the channel we must taken into account the increased velocity on the channel axis. The velocity along the axis in the inlet section of the channel

$$u=\frac{u_0 h}{h-\delta^*}, \tag{47}$$

which follows from the law of conservation of matter. We will consider that magnetic induction is sufficiently high and that the boundary layers on opposite walls of the channel do not join. Then having substituted into (39) the velocity from (47), we get an expression for the friction coefficient:

$$\lambda\equiv\frac{8\tau_w}{\rho u_0^2}=\frac{8}{\left(1-\frac{u_0\delta^*}{\nu\,\mathrm{Re}_h}\right)^2}\left[\frac{d\frac{u_0\delta^*}{\nu}}{d\,\mathrm{Re}_x}\left(0{,}8+\frac{2{,}6}{1-\frac{u_0\delta^*}{\nu\,\mathrm{Re}_h}}\frac{u_0\delta^*}{\nu}\right)+\right.$$
$$\left.+N\frac{u_0\delta^*}{\nu}\left(1-\frac{u_0\delta^*}{\nu\,\mathrm{Re}_h}\right)\right]. \tag{48}$$

In the first approximation we can substitute into (28) the value $\frac{u_0\delta^*}{\nu}$ which corresponds to a plate, i.e., use the expression

$$\frac{u_*\delta^*}{\nu} = 0{,}525\left(\frac{0{,}29}{e^{n_1}N}\right)^{\frac{1}{n_1}}\left(1 - e^{-1{,}25 n_1 N \mathrm{Re}_x}\right)^{\frac{1}{n_1}},$$

which follows from (41) and (45). For the section of developed flow ($\delta^* \approx$ const) we have, from (48) and (41),

$$\lambda_{min} = 8N\frac{\frac{u_*\delta^*}{\nu}}{1 - \frac{u_*\delta^*}{\nu \mathrm{Re}_h}} = \frac{4{,}2Ne^{k\zeta_{max}}}{1 - \frac{0{,}525e^{k\zeta_{max}}}{\mathrm{Re}_h}}, \qquad (49)$$

where $k\zeta_{max}$ is determined from (44) (when b = 0). In Fig. 6 the friction coefficient calculated from (49) is compared with Likodis' experimental results [7].

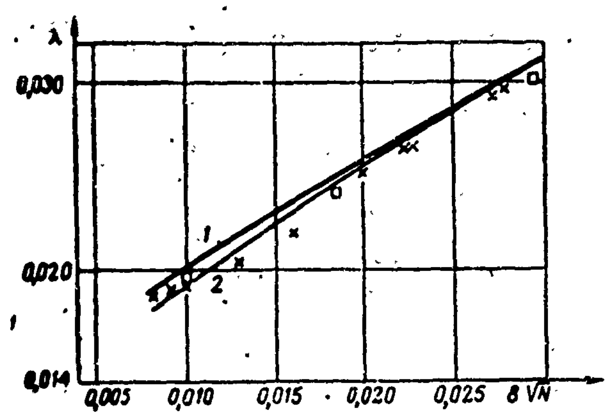

Fig. 6. Comparison of theoretical results for friction coefficient in an MHD channel with experimental data [7]. 1 – Re = $2.5 \cdot 10^4$; 2 – Re = $4.5 \cdot 10^4$. Squares: Re = $2.5 \cdot 10^4$; crosses: Re = $4.5 \cdot 10^4$.

The segment of possible comparison is bounded, along the abscissa, from below by the value of N for which δ_{max} = h and which can be determined approximately from Fig. 5, and from above by the value of N corresponding to the transition of turbulent flow into laminar. From the comparison it follows that (49) corresponds to experimental data within 5%.

Let us also note, as follows from (42), that if the velocity on the outer boundary of the boundary layer u and induction B are connected by the relationship

$$(1+\varkappa)\frac{du}{d\,\mathrm{Re}_x}=-\frac{\sigma B^2\nu}{\rho u_0},\qquad (50)$$

the solution of (42) can be written in the form

$$\int\limits_0^{\varkappa}\eta^2 e^{\varkappa}d\eta=\frac{0{,}29}{\varkappa}\int\limits_0^{\mathrm{Re}_x}\frac{u}{u_0}d\,\mathrm{Re}_x.$$

To satisfy condition (50) we set, e.g.,

$$u=u_0\mathrm{Re}_x^{-m},\ B=\sqrt{\frac{m(1+\varkappa)}{\sigma\nu}}\rho u_0\mathrm{Re}_x^{-\frac{m+1}{2}},\ 0\leqslant m<1.$$

Then we get

$$\int\limits_0^{\varkappa}\eta^2 e^{\varkappa}d\eta=\frac{0{,}29}{\varkappa}\frac{\mathrm{Re}_x^{-m+1}}{1-m}.$$

This method of solving the problem of the turbulent boundary layer with electrical and magnetic fields is also applicable during variable electrical conductivity and another dependence of u on x. In these latter cases the obtained differential equations should be solved numerically.

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